Rennes, January 11, 2002

Dear Jan,

Here is my computation of what could be called the "rigid cohomology groups" of $\overline{S}_0 := \mathrm{Spec}(O_{\overline{K}}/pO_{\overline{K}})$, where K is a complete discretely valued field of mixed characteristics $(0, p)$, with perfect residue field k. This is what I briefly told Berger in an e-mail last year, and what has led to the reference [3] in his thesis (*cf.* [Berger01, I.2, p. 19, l. 19]). I will use here Fontaine's notations, and I will give the dictionnary with Berger's notations at the end.

First, note that there is no a priori definition of such cohomology groups. Indeed, the construction of rigid cohomology for an algebraic variety X over k uses embeddings of X into smooth formal schemes over $W = W(k)$, and is based upon the de Rham cohomology of the generic fiber of such formal schemes. Thus, it is assumed that X is locally of finite type over k. Since this hypothesis is not satisfied by $\overline{S}_0$, we must provide an ad hoc definition.

One might think of writing $O_{\overline{K}}$ as the direct limit of $O_{K'}$ for finite sub-extensions K' of K, and try to define the rigid cohomology of $\overline{S}_0$ as $\varinjlim_{K'} H^*_{\mathrm{rig}}(\mathrm{Spec}(O_{K'}/p))$. That would only give the maximal unramified extension K_0^{nr} of $K_0 = \mathrm{Frac}(W(k))$. This is because rigid cohomology ignores infinitesimal deformations, hence $\mathbb{R}\Gamma_{\mathrm{rig}}(\mathrm{Spec}(O_{K'}/p)) = K'_0$ for all K', where $K'_0 = \mathrm{Frac}(W(k'))$, k' being the residue field of K'.

It seems to be a better idea to treat $\overline{S}_0$ as if it was a proper k-scheme, since it is at least universally closed over $\mathrm{Spec}(k)$. For a proper k-scheme X, there is a crystalline description of $\mathbb{R}\Gamma_{\mathrm{rig}}(X)$ which does not use embeddings into formal schemes, and still makes sense for arbitrary schemes. Namely, one can consider for every $m \geq 0$ the crystalline site of level m, defined as the usual crystalline site, except that thickenings carry a partial divided power structure of level m [Berthelot96] instead of a full divided power structure (the usual site is obtained for $m = 0$). Then there exists a canonical isomorphism

$$\mathbb{R}\Gamma_{\mathrm{rig}}(X) \xrightarrow{\sim} \mathbb{R}\varprojlim_m(\mathbb{R}\Gamma_{\mathrm{cris},(m)}(X/W) \otimes \mathbb{Q}).$$

So we can define the rigid cohomology of $\overline{S}_0$ by setting

$$\mathbb{R}\Gamma_{\mathrm{rig}}(\overline{S}_0) := \mathbb{R}\varprojlim_m(\mathbb{R}\Gamma_{\mathrm{cris},(m)}(\overline{S}_0/W) \otimes \mathbb{Q}).$$

With this definition, we get the following results :

a) There exists a canonical isomorphism

$$H^0_{\mathrm{rig}}(\overline{S}_0) \simeq B^+,$$

where B^+ is the period ring defined by Fontaine in [Fontaine82] ;

b) $H^n_{\mathrm{rig}}(\overline{S}_0) = 0$ for all $n > 0$.

To prove these, one first observes that the computation of $\mathbb{R}\Gamma_{\text{cris},(m)}(\overline{S}_0/W)$ can be done exactly as in the case $m = 0$: if $R = \varprojlim_F O_{\overline{K}}/p$ is Fontaine's ring, then, for all i, the W_i-scheme $\mathrm{Spec}(W_i(R))$ is formally étale over $\mathrm{Spec}(W_i)$; thus, the canonical surjection $W_i(R) \to O_{\overline{K}}/p$ defines a closed immersion such that the corresponding PD-neighbourhood is a final object in the crystalline site. It follows that $H^0_{\text{cris},(m)}(\overline{S}_0/W_i) = P_{(m)}(J_i)$, where $P_{(m)}(J_i)$ is the divided power envelope of level m of $J_i = \mathrm{Ker}(W_i(R) \to O_{\overline{K}}/p)$, while $H^n_{\text{cris},(m)}(\overline{S}_0/W_i) = 0$ for $n \geq 1$.

The classical map $\theta : W(R) \to O_C$ is a lifting of $W_i(R) \to O_{\overline{K}}/p$. Let $\theta_i : W_i(R) \to O_{\overline{K}}/p^i O_{\overline{K}}$ be the reduction of θ mod p^i, $I = \mathrm{Ker}(\theta)$, $I_i = \mathrm{Ker}(\theta_i)$. Note that $J_i = I_i + pW_i(R)$. Thanks to the compatibility with the divided powers of p imposed in the construction of the divided power envelopes $P_{(m)}$, we have

$$P_{(m)}(J_i) \;=\; P_{(m)}(I_i) \;=\; P_{(m)}(I)/p^i P_{(m)}(I).$$

Since $\mathbb{R}\Gamma_{\text{cris},(m)}(\overline{S}_0/W) \simeq \mathbb{R}\varprojlim_i \mathbb{R}\Gamma_{\text{cris},(m)}(\overline{S}_0/W_i)$, it follows that $H^0_{\text{cris},(m)}(\overline{S}_0/W) = \varprojlim_i P_{(m)}(J_i)$ is equal to the p-adic completion $\widehat{P}_{(m)}(I)$ of $P_{(m)}(I)$, and $H^n_{\text{cris},(m)}(\overline{S}_0/W) = 0$ for $n \geq 1$.

Let $A_{\text{cris},(m)} = \widehat{P}_{(m)}(I)$, $B^+_{\text{cris},(m)} = \widehat{P}_{(m)}(I) \otimes \mathbb{Q}$ (so that $A_{\text{cris},(0)} = A_{\text{cris}}$, $B^+_{\text{cris},(0)} = B^+_{\text{cris}}$ with Fontaine's notations [Fontaine94]). We want to check now that the transition maps $A_{\text{cris},(m+1)} \to A_{\text{cris},(m)}$ are injective. By construction, $P_{(m)}(I) = P_{(0)}(I^{(p^m)}) = P_{(0)}(I^{(p^m)} + pW(R))$, where $I^{(p^m)}$ is the ideal generated by the p^m-th powers of elements of I. Therefore, the m-th iterate ϕ^m of the Frobenius automorphism of $W(R)$ induces a semi-linear isomorphism $A_{\text{cris},(0)} \xrightarrow{\sim} A_{\text{cris},(m)}$. Thus the claim results from the fact that ϕ is injective on A_{cris} (in fact, it could also be checked directly using the structure theorem [Berthelot96, Prop. 1.5.3] since, for all i, I_i is generated by a regular element, and $W_i(R)/I_i$ is flat over W_i). We also get that the canonical homomorphism $A_{\text{cris},(m)} \to A_{\text{cris}}$ is an isomorphism on the image of ϕ^m inside A_{cris}.

Using this identification to work inside B^+_{cris}, statements a) and b) are reduced to :

a') $\bigcap_m B^+_{\text{cris},(m)} = B^+$;

b') $R^1 \varprojlim_m B^+_{\text{cris},(m)} = 0$.

Let me now recall Fontaine's definition of the ring B^+. For any ideal $\mathfrak{a} \subset R$, let $S_{\mathfrak{a}} \subset W(R) \otimes \mathbb{Q}$ be the subring of elements of the form $\sum_{n \gg -\infty} p^n[v_n]$, where, for all $n < 0$, $v_n \in \mathfrak{a}^{-n}$ and $[v_n]$ is the Teichmüller representative of v_n, and let $\widehat{S}_{\mathfrak{a}}$ be its p-adic completion. Then Fontaine shows that, for $\mathfrak{a} \subset \mathfrak{a}'$, the map $\widehat{S}_{\mathfrak{a}} \to \widehat{S}_{\mathfrak{a}'}$ is injective, and defines $B^+ := \bigcap_{\mathfrak{a}} B^+_{\mathfrak{a}}$, where $B^+_{\mathfrak{a}} = \widehat{S}_{\mathfrak{a}} \otimes \mathbb{Q}$.

Let $u \in R$ be an element such that $u^{(0)} = -p$ (using the description of R as the set of families $(u^{(n)})_{n \geq 0}$, $u^{(n)} \in O_C$, such that $(u^{(n+1)})^p = u^{(n)}$). Then $\alpha := [u] + p$ is a generator of I, and the ideal $\mathfrak{a} := (u) \subset R$ is the set of elements of valuation ≥ 1. If $\mathfrak{a}_n = (u^{p^n})$, then $B^+ = \bigcap_n B^+_{\mathfrak{a}_n}$. On the other hand, it follows from [Fontaine82, 4.2] that $S_{\mathfrak{a}_n} = W(R)[\alpha^{p^n}/p]$. Now, for all $m \geq 0$, there is a natural map $W(R)[\alpha^{p^{m+1}}/p] \to P_{(m)}(I) = P_{(0)}((\alpha^{p^m}))$, sending $\alpha^{p^{m+1}}$ to $(p-1)!(\alpha^{p^m})^{[p]}$, and a natural map $P_{(m)}(I) \to W(R)[\alpha^{p^m}/p]$, sending $(\alpha^{p^m})^{[k]}$ to $(p^k/k!)(\alpha^{p^m}/p)^k$. These maps extend to the p-adic completions. If we view these rings as sub-rings of A_{cris}, it follows that

$$\widehat{S}_{\mathfrak{a}_{m+1}} \subset \widehat{P}_{(m)}(I) \subset \widehat{S}_{\mathfrak{a}_m},$$

which proves assertion a').

To prove b'), one can adapt an argument of Kiehl [Kiehl67]. For $m' \leq m$, let $\rho_{m',m} : B^+_{\mathrm{cris},(m)} \to B^+_{\mathrm{cris},(m')}$ be the canonical map. Define

$$C_m = \prod_{m' \leq m} B^+_{\mathrm{cris},(m')}, \qquad D_m = \prod_{m' < m} B^+_{\mathrm{cris},(m')}.$$

Let $\delta_m : B^+_{\mathrm{cris},(m)} \to C_m$ be the diagonal map defined by the $\rho_{m',m}$, let $\pi_m : C_m \to D_m$ be the projection, and let $\rho_m : C_m \to D_m$ be defined by $\rho_m(x)_{m'} = \rho_{m',m'+1}(x_{m'+1})$. The exact sequences

$$0 \to B^+_{\mathrm{cris},(m)} \xrightarrow{\delta_m} C_m \xrightarrow{\pi_m - \rho_m} D_m \to 0$$

define a $\varprojlim$-acyclic resolution of the projective system $(B^+_{\mathrm{cris},(m)})_{m \geq 0}$. Using this, b') is reduced to proving that, if one denotes $C = \varprojlim_m C_m \simeq \prod_m B^+_{\mathrm{cris},(m)}$, and $\rho = \varprojlim_m \rho_m : C \to C$, the map $\mathrm{Id} - \rho : C \to C$ is surjective.

For each m, we define a norm on the $\mathbb{Q}_p$-vector space $B^+_{\mathrm{cris},(m)}$ by setting, for $x \neq 0$,

$$v(x) = \sup\{n | x \in p^n A_{\mathrm{cris},(m)}\}, \qquad \|x\| = p^{-v(x)}.$$

This insures that $\|\rho_{m',m}\| \leq 1$ for all $m' \leq m$. Fix some real number $\eta < 1$, and let $E \subset C$ be the subspace of families (x_m) such that $\eta^{-m}\|x_m\| \to 0$ for $m \to \infty$. Endow E with the norm defined by $\|x\| = \sup_m \eta^{-m}\|x_m\|$. Then E is a Banach space, stable under ρ, and on which $\|\rho\| \leq \eta$. Thus $\mathrm{Id} - \rho$ is surjective on E.

Observe now that $W(R) \otimes \mathbb{Q}$ is dense in $B^+_{\mathrm{cris},(m)}$ for all m. Given any $x = (x_m) \in C$, this allows to construct inductively a sequence of elements $y = (y_m) \in C$ such that $y_0 = 0$, and

$$\|(x_m - y_m) + \rho_{m,m+1}(y_{m+1})\| \leq \eta^{2m}.$$

Then $x' = (x_m - y_m + \rho_{m,m+1}(y_{m+1}))$ belongs to E. Thus, there exists $z \in E$ such that $x' = z - \rho(z)$. We now have $x = (\mathrm{Id} - \rho)(y + z)$, which ends the proof.

Let me finally indicate the correspondance with Berger's notations ($\mathfrak{a}$ denotes the ideal of R used above) :

Fontaine	Berger
R	$\widetilde{\mathbf{E}}^+$
$W(R)$	$\widetilde{\mathbf{A}}^+$
$W(R) \otimes \mathbb{Q}$	$\widetilde{\mathbf{B}}^+$
$B^+_{\mathfrak{a}}$	$\widetilde{\mathbf{B}}^+_{\mathrm{max}}$
B^+	$\widetilde{\mathbf{B}}^+_{\mathrm{rig}}$

I don't know whether this remark is useful. At least, I hope it answers your question. Best regards,

Pierre

References

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